RESEARCH STATEMENT

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***Abstract*—My research explores the strong connection between the degrees of large cardinals and the forcings to reduce them as finely as one degree. I am more recently interested in a similar idea applied to the finite realm exploring worlds where we mute the creative ability of a prime number.**

I. Introduction

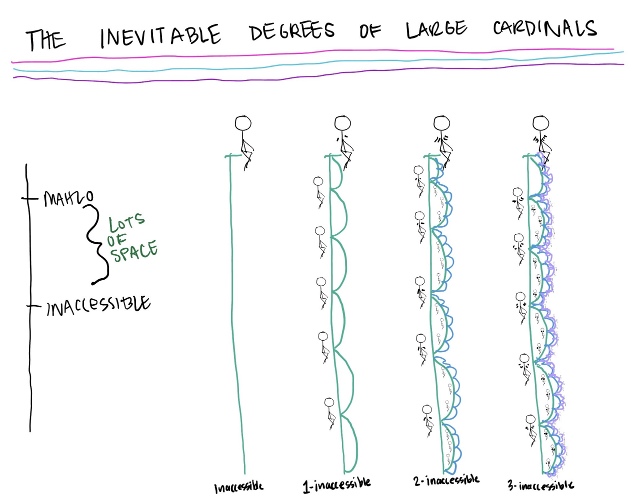
Almost every large cardinal studied so far comes with degrees of its existence, or seems to have the potential for degrees. For example, Mitchell rank for measurable cardinals provides degrees for measurable cardinals. I showed how to precisely define the higher degrees of inaccessible cardinals and Mahlo cardinals. Gitman and Habič showed how to define degrees of Ramsey cardinals. I also showed how to define a Mitchell rank for supercompact cardinals.

Along with the degrees of a large cardinal comes the forcings which can kill a large cardinal to any desired degree. It is not yet known if for every degree of a large cardinal there is a forcing to kill it so that in the forcing extension it has the desired maximum rank or degree, but so far they seem to go hand in hand. There are many killing-them-softly like results in set theory where in the forcing extension, a large cardinal's power has been reduced. For example:

Theorem 1 (Hamkins/Shelah): If κ is θ-supercompact where $\kappa < \theta$ and $\theta^{<\kappa} = \theta$, then there is a forcing extension where κ is θ-supercompact but not θ^+-supercompact.

Theorem 2 (Magidor): If κ is strongly compact, then there is a forcing extension where κ is strongly compact, but not supercompact.

Here is an image showing the inevitable degrees of inaccessible cardinals. A 1-inaccessible cardinal is a limit of inaccessible cardinals, a 2-inaccessible cardinal is a limit of 1-inaccessible cardinals and so on. We will see a result about softly killing inaccessible cardinals as well as a similar result for degrees of supercompact cardinals.



We extend the ideas of the killing-them-softly phenomenon in the degrees of large cardinals, and the general forcing and large cardinal genre of destruction and preservation, to the finite world and begin killing primes. We see what happens when we halt the creative ability of primes: we get new primes.

II. Killing degrees of large cardinals

There are many theorems about killing degrees of large cardinals softly. There are two parts to this area of research: defining the degrees of the large cardinals and creating the forcings which reduce the degree of a large cardinal to any desired level. In my PhD thesis, I defined the higher degrees of inaccessible cardinals and defined the Mitchell rank for supercompact cardinals, analogous to Mitchell rank for measurable cardinals. Let us see the definitions of degrees of inaccessible cardinals and supercompact cardinals, but there are many more degrees of large cardinals and accompanying killing-them-softly theorems.

1) Inaccessible Cardinals: An uncountable cardinal κ is *inaccessible* if and only if κ is a regular, strong limit cardinal. A cardinal κ is *1-inaccessible* if and only if κ is a limit of inaccessible cardinals. A cardinal κ is *α-inaccessible* if and only if κ is inaccessible, and for every $\beta < \alpha$, the cardinal κ is a limit of β-inaccessible cardinals. If κ is κ-inaccessible, then κ is called *hyper-inaccessible*. Once we get to this level, we need to relativize because there cannot be a $\kappa + 1$-inaccessible cardinal even though the degrees keep going. To get to the next degree, use the symbol Ω, the symbol for the class of ordinals, so that κ is κ-inaccessible if and only if κ is Ω-inaccessible. Then we can define a *1-Ω-inaccessible* cardinal κ to be a limit of Ω-inaccessible cardinals. Continue in this way until we get to a cardinal κ which is $\Omega \cdot \kappa$-inaccessible (i.e. hyper$^\kappa$-inaccessible). To go to the next step, define κ to be *richly-inaccessible* if and only if κ is Ω^2-inaccessible. Here are a few of the first few classes of inaccessible cardinals described with this notation system:

κ is Ω-inaccessible $\iff$ κ is hyper-inaccessible

κ is Ω^2-inaccessible $\iff$ κ is richly-inaccessible

κ is Ω^3-inaccessible $\iff$ κ is utterly-inaccessible

κ is $\Omega^3 \cdot 2$-inaccessible $\iff$ κ is utterly^2-inaccessible

κ is Ω^4-inaccessible $\iff$ κ is deeply-inaccessible

κ is Ω^5-inaccessible $\iff$ κ is truly-inaccessible

κ is Ω^6-inaccessible $\iff$ κ is eternally-inaccessible

κ is Ω^7-inaccessible $\iff$ κ is vastly-inaccessible

κ is $\Omega^7 + \Omega^6 + \Omega^5 + \Omega^4 + \Omega^3 + \Omega^2 + \Omega + \alpha$-inaccessible $\iff$ κ is hyper-richly-utterly-deeply-truly-eternally-vastly-inaccessible

This notation system is for meta-ordinals. This system is like Cantor's normal form for ordinals, but instead of ω we use Ω, a symbol for the order-type of Ord. The degree of any inaccessible cardinal is denoted by t, a formal syntactic expression for a meta-ordinal of the form $\Omega^\alpha \cdot \beta$ + $\Omega^\eta \cdot \gamma + \cdots + \Omega \cdot \delta + \sigma$ where $\beta, \gamma, \delta, \sigma \in$ Ord. If κ is t-inaccessible, then the only restriction on the meta-ordinal t is that all of the ordinals in t are less than or equal to κ. In this way, inaccessible cardinals with the same degree of inaccessibility can be described with the same meta-ordinal.

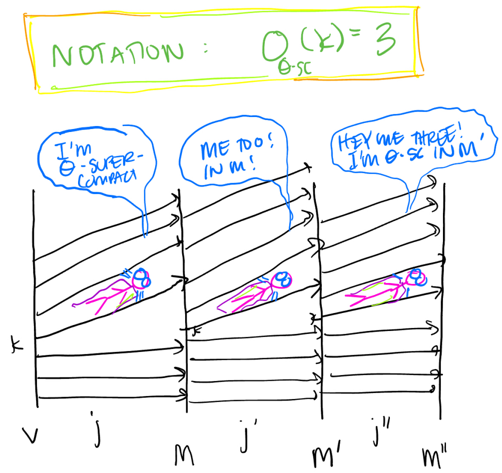
The most general theorem for killing degrees of inaccessible cardinal is the following.

Theorem 3: If κ is t-inaccessible, where t is a meta-ordinal term having parameters less than κ, then there is a forcing extension $V[G]$ where κ is t-inaccessible but not $t + 1$-inaccessible.

In the proof, we first force the GCH if needed, then force to add a club C which contains no t-inaccessible cardinals, and then use Easton forcing to kill all the strong limits that are not limit points of C so that κ is no longer $t + 1$-inaccessible (but we show that κ is still t-inaccessible by a density argument).

2) Supercompact Cardinals : An uncountable cardinal κ is *θ-supercompact* if and only if there is an elementary embedding $j : V \to M$, with critical point κ and $M^\theta \subseteq M$, where $\kappa < \theta < j(\kappa)$. Equivalently, κ is θ-supercompact if and only if there is a normal fine measure on $P_\kappa(\theta)$. A cardinal κ is *supercompact* if and only if it is θ-supercompact for every $\theta > \kappa$.

To see degrees of κ a θ-supercompact cardinal, with witness $j : V \to M$, look inside M and ask if κ is θ-supercompact there.



As for measurable cardinals, an analogous rank to Mitchell rank can be assigned to supercompactness embeddings. Suppose a cardinal κ is θ-supercompact for fixed θ. If μ and ν are normal fine measures on $P_\kappa\theta$, define the Mitchell relation $\mu \lhd_{\theta-sc} \nu$ if and only if $\mu \in M_\nu$ where $j : V \to M_\nu$ is an ultrapower embedding by ν. Since the relation $\lhd_{\theta-sc}$ is well-founded for given κ and θ, the Mitchell rank of μ is its rank with respect to $\lhd_{\theta-sc}$. Thus for fixed κ and θ, the notation $o_{\theta-sc}(\kappa) = \alpha$ means the height of $\lhd_{\theta-sc}$ on normal fine measures on $P_\kappa\theta$ is α. By definition, $o_{\theta-sc}(\kappa)$ is the height of the well-founded Mitchell relation $\lhd_{\theta-sc}$ on $m(\kappa)$, the collection of normal fine measures on $P_\kappa\theta$. Thus, the definition of rank for $\mu \in m(\kappa)$ with respect to $\lhd_{\theta-sc}$ is $o_{\theta-sc}(\mu) = \sup\{o_{\theta-sc}(\nu) + 1 \mid \nu \lhd_{\theta-sc} \mu\}$. Thus $o_{\theta-sc}(\kappa) = \sup\{o_{\theta-sc}(\mu) + 1 \mid \mu \in m(\kappa)\}$. So that $o_{\theta-sc}(\kappa) = 0$ means that κ is not θ-supercompact, and $o_{\theta-sc}(\kappa) > 0$ means that κ is θ-supercompact.

The most general theorem considers the case of softly killing Mitchell rank for θ-supercompactness for θ-supercompact cardinals κ when $\theta > \kappa^+$.

Theorem 4: For any $V \models ZFC + GCH$, any $\Theta : ORD \to ORD$ and any $F : ORD \to ORD$, there is a forcing extension $V[G]$ where, if κ is $\Theta(\kappa)$-supercompact, $\Theta \restriction \kappa$ represents $\Theta(\kappa)$, $F \restriction \kappa$ represents $F(\kappa)$ in V, and $\Theta''\kappa \subseteq \kappa$, then $o_{\Theta(\kappa)-sc}(\kappa)^{V[G]} = \min\{o_{\Theta(\kappa)-sc}(\kappa)^V, F(\kappa)\}$.

I gave a talk at the recent APA/ASL meeting in Chicago and gave a graphic novel-like proof outline for this theorem:

III. KILLING PRIMES

What makes primes so special and intriguing? Prime numbers have two major features: they are *creative* and they are *unique*. Primes are unique because a prime number p has only two positive integer factors, namely 1 and p. Primes are also very creative. The fundamental theorem of arithmetic says that every positive integer can be written as a product of primes. Thus, the creativity of the primes gives us of all the rest of the natural numbers. In M, we see what happens when we pause or mute the creative ability of a prime number.

In M, instead of numbers, there are structures made of prime blocks. M is constructed in such a way that the prime numbers of V are the prime blocks of M.

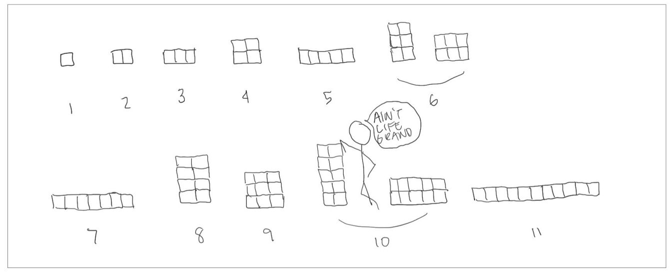
In M, we have the absence of blocks, the nothing block, which we interpret as 0. So, the nothing block in M is interpreted as 0 in our world. Then, in M, we are given □, which we would interpret as 1 in V. Just like in our world, we will not think of □ as a prime block. In M, we can only build with prime blocks. And all number structures will have a rectangular shape. Next, we are given

which is the 2-block in M, which we interpret as 2 in V. Since we cannot make a 3-block from the 2-block, we need a new prime block at the next stage, the 3-block.

We can use the 2-blocks to make 4.

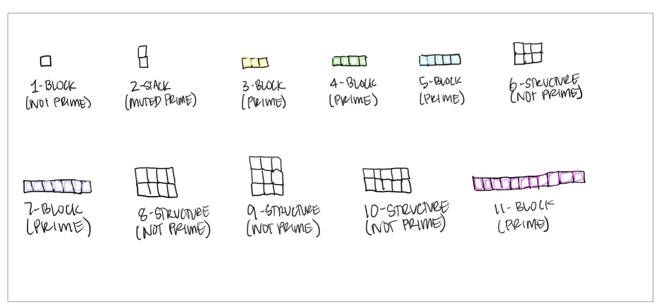
And this is how all of the number structures look in M. They are rectangular shapes with a prime base. The prime base of the 4-structure is the 2-block. Since we can't make a rectangular shape for 5 with either the 2-blocks or the 3-blocks, we will need to ask for a new prime block. At any stage, if we cannot make a rectangular shape for the number structure from previous prime blocks, we can ask for a new prime block.

Here are the structures in M for 1 through 11.



What happens when 2 cannot be used as a prime base? Will another prime block appear? Looking at this extension through the eyes of V, everything will seem the same; we still have all of the natural numbers and the extension is still closed under addition and multiplication (and gives the correct result). Let us flip the 2-block vertically so that it can no longer be used as a prime block, let us call it a 2-stack.

Let us see what is going on in an extension of M, a world called M^{-2}, where there is no 2-block. Besides that the stable 2-block is now the unstable 2-stack, the main difference between M and the extension is that when we get to stage 4, we can no longer use the 2-blocks to build so we have to ask for a new prime block at this stage. Thus in M^{-2}, the number structure for 4 is a prime block, the 4-block.



Here are some of the theorems about killing primes so far:

Theorem 5: If p is prime in M, then there is an extension M^{-p} where p is no longer prime, and $M^{-p} \models p^2$ is prime. In addition, all other prime numbers are preserved in M^{-p}, and there are no other new primes.

Theorem 6: If S is a set of n many primes in M, then in an extension of M where exactly the primes in S are killed, there will be $2^n - 1$ many new primes.

Theorem 7: If P is the set of primes in M, then in the extension M^{-P} where all of these primes are killed, there are still infinitely many prime numbers (and infinitely many composite numbers). In particular, the new primes are previous composites which are either of the form p^2, where p is prime in V, or have only powers of primes exactly one in their prime factorization.

Theorem 8: There is an extension of M where a prime number amount of primes from M are killed and the cardinality of the number of new primes is also prime.

Theorem 9: If there are infinitely many primes of the form of $p^2 - 2$, then there is an extension of M where the twin prime conjecture is true.

Theorem 10: If there are infinitely many twin prime pairs (p, q) in M such that one of the pair $(p^2 - 2, q^2 - 2)$ is prime in M, then there is an extension of M where infinitely many twin primes are killed, but for each one killed there is at least one more twin prime pair in the extension so that in the extension there are still infinitely many twin prime pairs.

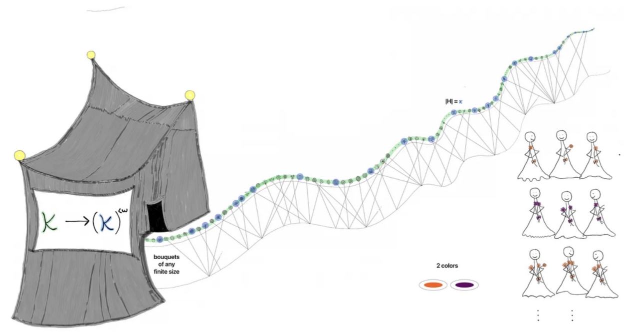
Theorem 11: If, in M, there is a very large even number e such that e is not the sum of two primes in M, then there is an extension N of M, where e is the sum of two primes in N.

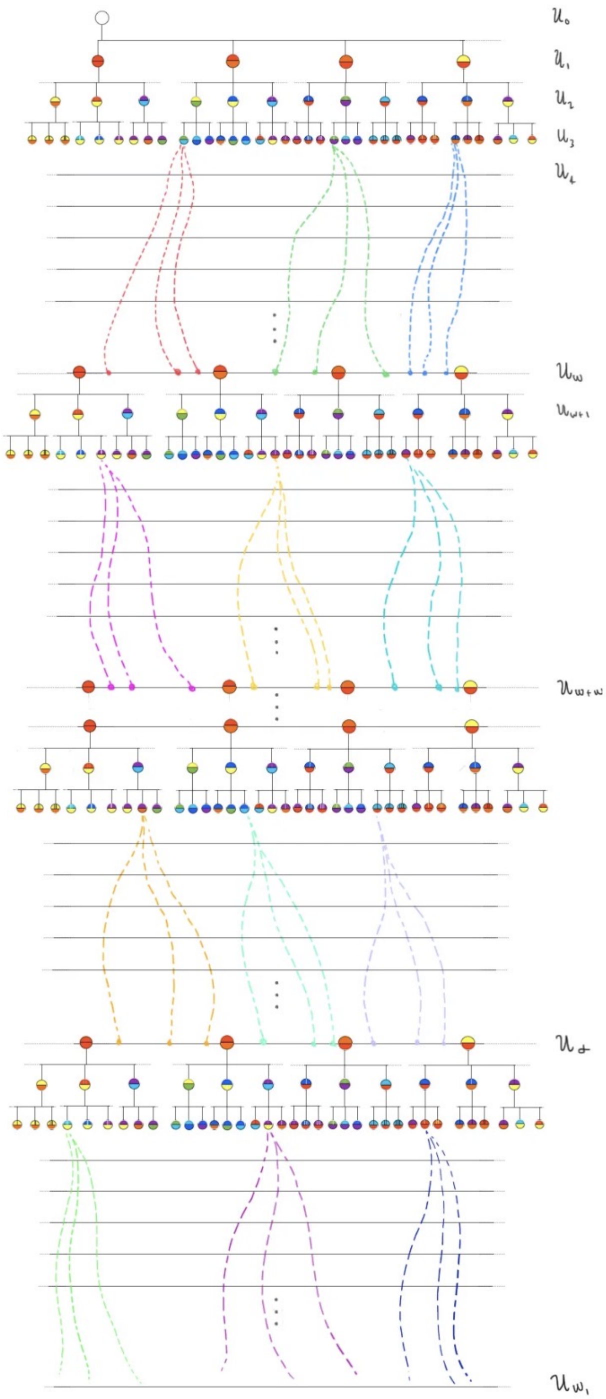
IV. Where do we go from here

Here are some future research projects and interests:

- A main interest is the targets of extendible cardinals - which I like to call abiding cardinals. A cardinal κ is an *abiding cardinal* if there is an $\eta < \kappa$ such that η is extendible with target κ. I would like to see what happens if we define degrees of abiding cardinals by counting the number of extendible cardinals of which it is the target.
- Another topic I would like to focus on is Vopěnka's principle. Vopěnka's principle says that if there is a class of models in the same language, there is a pair of models where one can be elementarily embedded in the other. Vopěnka's principle is a large cardinal notion and if the principle holds then an extendible cardinal exists. Vopěnka's principle is philosophical and romantic in nature. I would like to see if there are degrees of the principle and how Vopěnka's principle interacts with forcing.
- I would like to focus on forcing and the philosophy of forcing especially as it pertains to the killing of individual degrees of large cardinals. I would like make a list and categorize the forcings used to softly kill large cardinals. I would like to make a clean sweep of all of the known large cardinals that have degrees and see if we can force to gently kill them. I would like to compile a list of all theorems, large cardinals, and forcings that fit the theme of precisely making a cut in the large cardinal hierarchy with forcing.
- I would like to write about Ω philosophically and historically, and discuss the large cardinal properties of Ω and see if we can somehow force to softly kill its large cardinal properties.

- I would like to finish my book *The Set Theory Guide for Artists*. There are 7 chapters total: Infinity, The Axioms, Set Formation, Large Cardinals, Forcing, Independence of CH, Forcing and Large Cardinals. I am currently in Chapter 4 (posted on substack as it is written). Here are some of the images from the book. The first is an Aronszajn tree, and the second is a Ramsey cardinal.





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